Assignment 3

Hand in no. 1. 5, and 7 by Oct 2, 2018.

1. Let f be a function defined on (a, b) and $x_0 \in (a, b)$.
   (a) Show that f is Lipschitz continuous at x_0 if its left and right derivatives exist at x_0.
   (b) Construct a function Lipschitz continuous at x_0 whose one sided derivatives do not exist.

2. Optional. Let f be a function defined on $(a, b]$ which is integrable on $[c, b]$ for all $c \in (a, b)$. It is called improperly integrable over $(a, b]$ if

$$\lim_{c\to a^+} \int_c^b |f|$$

exists. When this happens,

$$\lim_{c\to a^+} \int_c^b f$$

also exists and we define the improper integral of f over $(a, b]$ to be

$$\int_a^b f = \lim_{c\to a^+} \int_c^b f \ .$$

   (a) Show that if f is integrable on $[a, b]$, its improper integral also exists and is equal to it usual integral.
   (b) Show that Riemann-Lebesgue Lemma holds for improperly integrable functions.

3. Optional. Show that

$$-\log\left|2\sin\frac{x}{2}\right| \sim \cos x + \frac{\cos 2x}{2} + \frac{\cos 3x}{3} + \cdots \ .$$

Suggestion. Verify this function is 2π-periodic and improperly integrable first. The calculation of a_0 is tricky, involving the definite integral $I = \int_0^{\pi/2} \log\sin t dt$. To evaluate it use $\sin t = 2\sin t/2 \cos t/2$ and eventually show $I = -\frac{\pi}{2}\log 2$.

4. Let a_n, b_n be the Fourier coefficients of some $f \in R_{2\pi}$.
   (a) Show that for each $r \in [0, 1)$, the trigonometric series given by

$$a_0 + \sum_{k=1}^{\infty} r^n (a_n \cos nx + b_n \sin nx)$$

   is uniformly convergent to some function in $C_{2\pi}$. Denote this function by $f_r(x)$.
   (b) Show that

$$f_r(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(z) f(x+z) dz,$$

   where the **Poisson kernel** P_r is given by

$$P_r(z) = \frac{1 - r^2}{1 - 2r\cos z + r^2} \ .$$

(c) Let f be continuous at x. Show that $\lim_{r\to 1} f_r(x) = f(x)$.

The treatment is parallel to that for the Dirichlet kernel (the parameter n is now replaced by r), but differs at the final step; we do not need Lipschitz continuity. Think about it.

5. (a) Can you find a cosine series which converges uniformly to the sine function on $[0, \pi]$? If yes, find one.
   (b) Is the series in (a) unique?
   (c) Can you find a cosine series which converges pointwisely to the sine function on $[-a, \pi]$ where a is a number in $(0, \pi)$?

6. Let f be an integrable function on $[-\pi, \pi]$. Show that for each $\varepsilon > 0$, there exists a trigonometric polynomial p satisfying $p < f$ on $[-\pi, \pi]$ and

$$\int_{-\pi}^{\pi} |f - p| < \varepsilon \ .$$

7. Show that there is a countable subset of $C[a, b]$ such that for each $f \in C[a, b]$, there is some $\varepsilon > 0$ such that $\|f - g\|_\infty < \varepsilon$ for some g in this set. Suggestion: Take this set to be the collection of all polynomials whose coefficients are rational numbers.

8. Let f be continuously on $[a, b] \times [c, d]$. Show that for each $\varepsilon > 0$, there exists a polynomial $p = p(x, y)$ so that

$$\left\|f - p\right\|_\infty < \varepsilon, \quad \text{in } [a, b] \times [c, d].$$

In fact, this result holds in arbitrary dimension.